Direct Proof

A proof must work for *all* cases at once. Checking examples is not a proof; algebra is, because a letter stands for every integer simultaneously.

Fact (Representing integers) — For integer n:

an even number	$2n$
an odd number	$2n+1$ (or $2n-1$)
consecutive integers	$n,\ n+1,\ n+2$
consecutive even numbers	$2n,\ 2n+2$
consecutive odd numbers	$2n-1,\ 2n+1$
a multiple of k	kn

To prove a result is even, aim for $2\times$(integer); a multiple of 8, aim for $8\times$(integer).

Example

Prove that $(2n+1)^2-(2n+1)$ is even for all positive integers n.

$(2n+1)^2-(2n+1)=(2n+1)\Big[(2n+1)-1\Big]=2n(2n+1)$, *which is* $2\times$ *integer.*
(Factorising is quicker than expanding.)

Example

Prove that the difference between the squares of two consecutive odd numbers is divisible by 8.

$(2n+1)^2-(2n-1)^2=(4n^2+4n+1)-(4n^2-4n+1)=8n$

Example
Prove that the square of any odd number is one more than a multiple of 8.

$(2n+1)^2 = 4n^2+4n+1 = 4n(n+1)+1$
n and $n+1$ are consecutive, so one of them is even: $n(n+1) = 2k$.
Hence $(2n+1)^2 = 8k+1$.

Example

1. Show that $(n-1)n(n+1) = n^3-n$.
2. Hence prove that n^3-n is divisible by 6 for every integer n.

1. $(n-1)(n+1) = n^2-1$, so $(n-1)n(n+1) = n(n^2-1) = n^3-n$.
2. Of any three consecutive integers, at least one is even and exactly one is a multiple of 3. So the product is divisible by both 2 and 3, hence by 6.

Example (Multiplying three brackets)
Expand and simplify $(x+1)(x+2)(x+3)$.

$(x+1)(x+2) = x^2+3x+2$
$(x^2+3x+2)(x+3) = x^3+3x^2+3x^2+9x+2x+6 = x^3+6x^2+11x+6$

Textbook Exercises: SPS Course 0.1, Exercise 2

Disproof and Implication

Definition. A statement of the form "for all n, ..." is disproved by a single counterexample.

Example

Disprove: "$n^2 + n + 41$ is prime for every positive integer n."

$n = 40$: $40^2 + 40 + 41 = 40(40 + 1) + 41 = 41 \times 41$. *Not prime.*

Definition.

$$P \Rightarrow Q \text{ (P implies Q)} \qquad P \Leftarrow Q \qquad P \iff Q \text{ (P if and only if Q)}$$

Proving $P \Rightarrow Q$ says nothing about the converse $Q \Rightarrow P$.

Example

Insert $\Rightarrow$, $\Leftarrow$ or $\iff$ between each pair:

1. $x = 3$ $\qquad x^2 = 9$
2. $5x - 3 = 27$ $\qquad x = 6$
3. n is divisible by 4 $\qquad n^2$ is divisible by 4

1. $\Rightarrow$ *(x could be -3)* *2.* $\iff$ *3.* $\Rightarrow$ *($n = 6$: 36 is divisible by 4, 6 is not)*

Proof by Factorisation

Example

Prove that $n^2 - 1$ is never prime for any integer $n > 2$.

$n^2 - 1 = (n-1)(n+1)$. *For* $n > 2$ *both factors exceed* 1, *so* $n^2 - 1$ *is a product of two smaller integers: not prime.*

Example

Find all pairs of positive integers x, y with $x^2 - y^2 = 35$, proving you have found them all.

$(x-y)(x+y) = 35$ *with* $0 < x-y < x+y$, *so the factor pairs are* 1×35 *and* 5×7.
$x - y = 1,\ x + y = 35 \implies (18, 17) \qquad x - y = 5,\ x + y = 7 \implies (6, 1)$
Every solution gives a factor pair, so these are the only two.

Example

Prove that two non-zero squares can never differ by 2.

Suppose $x^2 - y^2 = 2$ *with* x, y *positive integers. Then* $(x-y)(x+y) = 2$, *so* $x - y = 1$ *and* $x + y = 2$, *giving* $x = \frac{3}{2}$ *— not an integer. No such squares exist.*

Exercise. Find the last digit of $2^{333} + 7^{333}$. (Track the cycle of last digits of the powers of 2 and of 7.)

Exercise. Prove that the sum of any four consecutive integers is never divisible by 4.

Textbook Exercises: SPS Course 0.1, Exercises 1, 5B and Revision Exercise 0.1